Likelihood calculations for vsn

Wolfgang Huber

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1 Introduction

This vignette contains the computations that underlie the numerical code of vsn. If you are a new user and looking for an introduction on how to use vsn, please refer to the vignette Robust calibration and variance stabilization with vsn, which is provided separately.

2 Setup and Notation

Consider the model
\[ \text{arsinh} \left( f(b_i) \cdot y_{ki} + a_i \right) = \mu_k + \varepsilon_{ki} \] where \( \mu_k \), for \( k = 1, \ldots, n \), and \( a_i, b_i \), for \( i = 1, \ldots, d \) are real-valued parameters, \( f \) is a function \( \mathbb{R} \to \mathbb{R} \) (see below), and \( \varepsilon_{ki} \) are i.i.d. Normal with mean 0 and variance \( \sigma^2 \). \( y_{ki} \) are the data. In applications to \( \mu \)-array data, \( k \) indexes the features and \( i \) the arrays and/or colour channels.

Examples for \( f \) are \( f(b) = b \) and \( f(b) = e^b \). The former is the most obvious choice; in that case we will usually need to require \( b_i > 0 \). The choice \( f(b) = e^b \) assures that the factor in front of \( y_{ki} \) is positive for all \( b \in \mathbb{R} \), and as it turns out, simplifies some of the computations.

In the following calculations, I will also use the notation
\[ Y \equiv Y(y, a, b) = f(b) \cdot y + a \]
\[ h \equiv h(y, a, b) = \text{arsinh} \left( f(b) \cdot y + a \right) \]
The probability of the data \((y_{ki})_{k=1\ldots n, i=1\ldots d}\) lying in a certain volume element of \(y\)-space (hyperrectangle with sides \([y_{ki}^\alpha, y_{ki}^\beta]\)) is

\[
P = \prod_{k=1}^{n} \prod_{i=1}^{d} \int_{y_{ki}^\alpha}^{y_{ki}^\beta} dy_{ki} \ p_{\text{Normal}}(h(y_{ki}), \mu_k, \sigma^2) \ \frac{dh}{dy}(y_{ki}),
\]

where \(\mu_k\) is the expectation value for feature \(k\) and \(\sigma^2\) the variance.

With

\[
p_{\text{Normal}}(x, \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)
\]

the likelihood is

\[
L = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^{nd} \prod_{k=1}^{n} \prod_{i=1}^{d} \exp\left(-\frac{(h(y_{ki}) - \mu_k)^2}{2\sigma^2}\right) \cdot \frac{dh}{dy}(y_{ki}).
\]

For the following, I will need the derivatives

\[
\frac{\partial Y}{\partial a} = 1
\]

\[
\frac{\partial Y}{\partial b} = y \cdot f'(b)
\]

\[
\frac{dh}{dy} = \frac{f(b)}{\sqrt{1 + (f(b)y + a)^2}} = \frac{f(b)}{\sqrt{1 + Y^2}}
\]

\[
\frac{\partial h}{\partial a} = \frac{1}{\sqrt{1 + Y^2}},
\]

\[
\frac{\partial h}{\partial b} = \frac{y}{\sqrt{1 + Y^2}} \cdot f'(b).
\]

Note that for \(f(b) = b\), we have \(f'(b) = 1\), and for \(f(b) = e^b\), \(f'(b) = f(b) = e^b\).

## 3 Likelihood for Incremental Normalization

Here, \textit{incremental normalization} means that the model parameters \(\mu_1, \ldots, \mu_n\) and \(\sigma^2\) are already known from a fit to a previous set of \(\mu\)-arrays, i.e. a set of reference arrays. See Section 4 for the profile likelihood approach that is used if \(\mu_1, \ldots, \mu_n\) and \(\sigma^2\) are not known and need to be estimated from the same data. Versions \(\geq 2.0\) of the \textit{vsn} package implement both of these approaches; in versions \(1.X\) only the profile likelihood approach was implemented, and it was described in the initial publication \[1\].
First, let us note that the likelihood is simply a product of independent terms for different $i$. We can optimize the parameters $(a_i, b_i)$ separately for each $i = 1, \ldots, d$. From the likelihood we get the $i$-th negative log-likelihood

$$-\log(L) = \sum_{i=1}^{d} -LL_i$$

$$-LL_i = \frac{n}{2} \log (2\pi\sigma^2) + \frac{n}{2} \log \left( \frac{(h(y_{ki}) - \mu_k)^2}{2\sigma^2} + \log \frac{1 + Y_{ki}^2}{f(b_i)} \right)$$

$$= \frac{n}{2} \log (2\pi\sigma^2) - n \log f(b_i) + \sum_{k=1}^{n} \left( \frac{(h(y_{ki}) - \mu_k)^2}{2\sigma^2} + \frac{1}{2} \log (1 + Y_{ki}^2) \right)$$

This is what we want to optimize as a function of $a_i$ and $b_i$. The optimizer benefits from the derivatives. The derivative with respect to $a_i$ is

$$\frac{\partial}{\partial a_i} (-LL_i) = \sum_{k=1}^{n} \left( \frac{r_{ki}}{\sigma^2} + A_{ki}Y_{ki} \right) A_{ki}$$

and with respect to $b_i$

$$\frac{\partial}{\partial b_i} (-LL_i) = -n \frac{f'(b_i)}{f(b_i)} + \sum_{k=1}^{n} \left( \frac{r_{ki}}{\sigma^2} + A_{ki}Y_{ki} \right) A_{ki}y_{ki}$$

Here, I have introduced the following shorthand notation for the “intermediate results” terms

$$r_{ki} = h(y_{ki}) - \mu_k$$

$$A_{ki} = \frac{1}{\sqrt{1 + Y_{ki}^2}}.$$
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into the negative log-likelihood. The result is called the negative profile log-likelihood

$$-PLL = \frac{nd}{2} \log (2\pi \hat{\sigma}^2) + \frac{nd}{2} - n \sum_{j=1}^{d} \log f(b_j) + \frac{1}{2} \sum_{k=1}^{n} \sum_{j=1}^{d} \log \sqrt{1 + Y_{kj}^2}. \tag{21}$$

Note that this no longer decomposes into a sum of terms for each \( j \) that are independent of each other – the terms for different \( j \) are coupled through Equations 19 and 20. We need the following derivatives.

\[
\frac{\partial \hat{\sigma}^2}{\partial a_i} = \frac{2}{n^d} \sum_{k=1}^{n} r_{ki} \frac{\partial h(y_{ki})}{\partial a_i} = \frac{2}{n^d} \sum_{k=1}^{n} r_{ki} A_{ki} \tag{22}
\]

\[
\frac{\partial \hat{\sigma}^2}{\partial b_i} = \frac{2}{n^d} \cdot f'(b_i) \sum_{k=1}^{n} r_{ki} A_{ki} y_{ki} \tag{23}
\]

So, finally

\[
\frac{\partial}{\partial a_i} (-PLL) = \frac{nd}{2\hat{\sigma}^2} \cdot \frac{\partial \hat{\sigma}^2}{\partial a_i} + \sum_{k=1}^{n} A_{ki}^2 Y_{ki} = \sum_{k=1}^{n} \left( \frac{r_{ki}}{\hat{\sigma}^2} + A_{ki} Y_{ki} \right) A_{ki} \tag{24}
\]

\[
\frac{\partial}{\partial b_i} (-PLL) = -n \left( \frac{f'(b_i)}{f(b_i)} + f'(b_i) \sum_{k=1}^{n} \left( \frac{r_{ki}}{\hat{\sigma}^2} + A_{ki} Y_{ki} \right) A_{ki} y_{ki} \right) \tag{25}
\]
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Likelihoods, from Equations 12 and 21:

\[-LL_i = \frac{n}{2} \log (2\pi \sigma^2) + \sum_{k=1}^{n} \frac{(h(y_{ki}) - \mu_k)^2}{2\sigma^2} + \sum_{k=1}^{n} -n \log f(b_i) + \frac{1}{2} \sum_{k=1}^{n} \log(1 + Y_{ki}^2)\]

\[-PLL = \frac{nd}{2} \log (2\pi \hat{\sigma}^2) + \frac{nd}{2} \sum_{k=1}^{d} \left( \frac{-n \log f(b_i) + \frac{1}{2} \sum_{k=1}^{n} \log(1 + Y_{ki}^2)}{\text{residuals}} \right)\]

The computations in the C code are organised into steps for computing the terms “scale”, “residuals” and “jacobian”.

Partial derivatives with respect to \(a_i\), from Equations 15 and 24:

\[\frac{\partial}{\partial a_i} (-LL_i) = \frac{n}{2} \sum_{k=1}^{n} \left( \frac{r_{ki}}{\sigma^2} + A_{ki} Y_{ki} \right) A_{ki} \]

\[\frac{\partial}{\partial a_i} (-PLL) = \frac{nd}{2} \sum_{k=1}^{d} \left( \frac{-n \log f(b_i) + \frac{1}{2} \sum_{k=1}^{n} \log(1 + Y_{ki}^2)}{\text{residuals}} \right) A_{ki} \]

Partial derivatives with respect to \(b_i\), from Equations 16 and 25:

\[\frac{\partial}{\partial b_i} (-LL_i) = -n \frac{f'(b_i)}{f(b_i)} + f'(b_i) \sum_{k=1}^{n} \left( \frac{r_{ki}}{\sigma^2} + A_{ki} Y_{ki} \right) A_{ki} y_{ki} \]

\[\frac{\partial}{\partial b_i} (-PLL) = -n \frac{f'(b_i)}{f(b_i)} + f'(b_i) \sum_{k=1}^{n} \left( \frac{r_{ki}}{\sigma^2} + A_{ki} Y_{ki} \right) A_{ki} y_{ki}. \]

Note that the terms have many similarities – this is used in the implementation in the C code.

References
